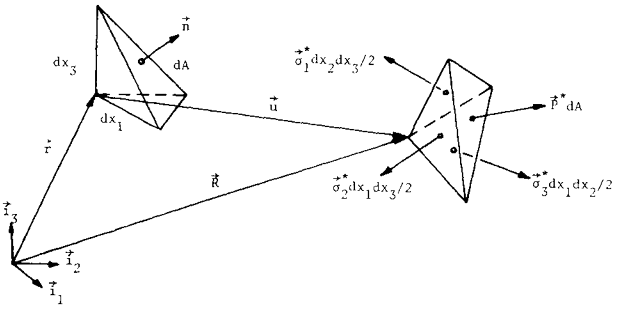

Fig. 6 Equilibrium with a surface traction.

$$\partial \boldsymbol{\sigma}_i^*/\partial x_i + \mathbf{X}^* = 0 \qquad \text{(A1)}$$

where $\boldsymbol{\sigma}_i^*$ is the stress vector referred to the undeformed ith face area and $\mathbf{X}^*$ is the body force vector referred to the undeformed volume. Resolving $\boldsymbol{\sigma}_i^*$ in the nonorthogonal lattice vector ($\mathbf{G}_j$) directions yields

$$\boldsymbol{\sigma}_i^* = \sigma_{ij}\mathbf{G}_j \qquad \text{(A2)}$$

where σ_{ij}^* are the Trefftz components of stress, referred to the undeformed ith face area, which can be shown to be symmetric.[1] Since $\mathbf{G}_j = (\partial \mathbf{R}/\partial x_j)$ where $\mathbf{R}$ is the final state position vector it is seen that§

$$\mathbf{G}_j = (\delta_{js} + \partial u_s/\partial x_j)\mathbf{i}_s \qquad \text{(A3)}$$

where $\mathbf{i}_s$ is the unit vector in the sth orthogonal Cartesian direction. Decomposing $\mathbf{X}^*$ into its $\mathbf{i}_s$ components

$$\mathbf{X}^* = X_s^*\mathbf{i}_s \qquad \text{(A4)}$$

§ Note that $\mathbf{R} = \mathbf{r} + \mathbf{u} = (X_s + u_s)\mathbf{i}_s$; therefore $(\partial \mathbf{R}/\partial x_j) = [\delta_{js} + (\partial u_s/\partial x_j)]\mathbf{i}_s$.

and putting the results of Eqs. (A2–A4) into Eq. (A1) yields the following scalar equations of equilibrium:

$$\frac{\partial}{\partial x_i}[(\delta_{js} + \partial u_s/\partial x_j)\sigma_{ij}^*] + X_s^* = 0 \qquad \text{(A5)}$$

Referring to Fig. 6, $\mathbf{p}^*$ is the prescribed traction referred to the undeformed oblique face, and noting that $dA(\mathbf{i}_i \cdot \mathbf{n}) = dA_i/2$, the equilibrium of the deformed tetrahedron is given by

$$\mathbf{p}^* = (\mathbf{i}_i \cdot \mathbf{n})\boldsymbol{\sigma}_i^* \equiv n_i \boldsymbol{\sigma}_i^* \qquad \text{(A6)}$$

and by using Eqs. (A2) and (A3) and defining the $\mathbf{i}_s$ components of $\mathbf{p}$ as p_s^*, (A6) becomes

$$p_s^* = \sigma_{ij}^* n_i[\delta_{js} + (\partial u_s/\partial x_j)] \qquad \text{(A7)}$$

Equations (A5) and (A7) are the desired nonlinear equations that describe the equilibrium condition and the surface tractions, respectively.

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Buckling of Polar Orthotropic Annular Plates under Uniform Internal Pressure

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The problem of buckling of polar orthotropic annular plates under uniform internal pressure is analyzed by the classical Rayleigh-Ritz method. Direct application of the method with simple polynomials as admissible functions is found to be inconvenient for large hole sizes, particularly, in the case of both edges clamped. In such situations, it is shown that the method in conjunction with a coordinate transformation introduced is convenient. Detailed numerical investigations have been carried out with regard to the convergence of solutions as well as the behavior of rounding errors in computations. Accurate estimates of critical buckling loads are obtained for various hole sizes and rigidity ratios and for all combinations of clamped, simply supported, and free edge conditions. A comparison with the results of isotropic plates has brought out some interesting features.

Nomenclature

a, b = radii of inner and outer edges, respectively
c = $a^2/(b^2 - a^2)$
C, S, F = clamped, simply supported, and free edge conditions, respectively
D = $Eh^3/12(1-\nu^2)$
D_r = $E_r h^3/12(1-\nu_r\nu_\theta)$
D_θ = $E_\theta h^3/12(1-\nu_r\nu_\theta)$
D_1 = $\nu_r D_\theta = \nu_\theta D_r$
E_r, E_θ = Young's moduli in radial and tangential directions, respectively
h = thickness of plate
k = $(E_\theta/E_r)^{1/2} = (D_\theta/D_r)^{1/2}$
p_i = uniform in-plane radial pressure at the inner edge
r, θ = polar coordinates of a point in midplane of plate
T = potential energy due to in-plane forces during bending
u_m, v_m = admissible functions
V = strain energy due to bending
$W(r)$ = lateral displacement
y = $(r^2-a^2)/(b^2-a^2)$
ν_r, ν_θ = Poisson's ratios
σ_r, σ_θ = normal stresses in radial and tangential directions, respectively
$\sigma_{r\theta}$ = shear stress

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1. Introduction

Analysis of polar orthotropic circular and annular plates based on the classical theory of thin plates is useful in predicting the elastic behavior of many structural components of practical importance, such as corrugated circular diaphragms as used in pressure capsules, perforated circular plates containing rectangular array of holes used as base plates of containers, and pressure vessels, circular plates stiffened with radial and/or circumferential ribs and plates fabricated out of modern laminated composites. The problem of elastic stability of these plates has received comparatively less attention than the problems of extension, bending, and vibration. There are, however, a few investigations reported in literature for polar orthotropic circular plates (without holes).[1-3] Woinowsky-Krieger[1] obtained some numerical results for axisymmetric buckling with simply supported and clamped edge conditions. Mossakowski[2] investigated the same problem in great detail and offered a general solution for axisymmetric as well as asymmetric modes. He has also given graphs and approximate solutions to facilitate the determination of buckling loads for a range of elastic constants. Pandalai and Patel[3] considered again the axisymmetric buckling and obtained some approximate values of critical buckling loads. In the case of annular plates, Vijayakumar and Joga Rao[4,5] used the classical Rayleigh-Ritz method for two cases of the annulus subjected to 1) equal pressures and 2) external pressure only. Assuming that the buckling mode corresponds to axisymmetric mode, estimates of critical buckling loads were obtained for various edge support conditions but for hole ratio of $\frac{1}{2}$ only.

It appears that the problem of buckling of polar orthotropic annular plates subjected to uniform internal pressure has not been considered till now. Even in the isotropic case, there is only one investigation[6] and that, too, for annular plates with clamped inner and free outer edges. In the abovementioned investigation,[6] an iterative procedure was used in the analysis, and critical buckling loads estimated at the second stage of the procedure were reported. In the present investigation critical buckling loads in the orthotropic case are estimated by the classical Rayleigh-Ritz method for all nine combinations of clamped, simply supported, and free edge conditions. The method with simple polynomials as admissible functions is found to be suitable for small hole sizes only. For large hole sizes, the method is used in conjunction with a coordinate transformation which was proposed earlier while considering free flexural vibrations of these plates,[7] and is found to yield accurate results. Extensive numerical data are obtained for various values of rigidity ratio and hole sizes and are presented in the form of tables and graphs. A comparison of results with those of isotropic plates has brought out some interesting features.

2. Mathematical Analysis

A thin annular plate of uniform thickness h subjected to uniform internal pressure p_i is considered (Fig. 1). The material

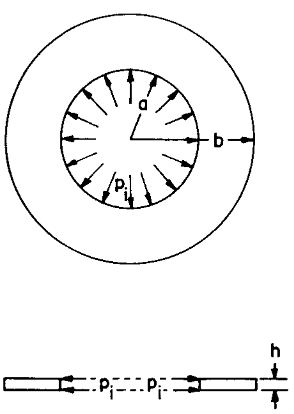

Fig. 1 Annular plate under uniform internal pressure.

of the plate is assumed to be homogeneous and cylindrically (polar) orthotropic. The prebuckling membrane stresses[8] are given by

$$\left.\begin{aligned}\sigma_r &= -\frac{p_i(a/r)^{k+1}}{1-(a/b)^{2k}}[1-(r/b)^{2k}]\\ \sigma_\theta &= \frac{p_i(a/r)^{k+1}k}{1-(a/b)^{2k}}[1+(r/b)^{2k}]\\ \text{and}\qquad \sigma_{r\theta} &= 0\end{aligned}\right\} \tag{1}$$

For $p_i > 0$, the hoop stress σ_θ is tensile and, hence, the plate always buckles symmetrically with respect to center of the plate.[6] In such a case, the strain energy V of bending and the potential energy T due to midplane forces are[4]

$$V = \pi\int_a^b\left[D_r\left(\frac{d^2W}{dr^2}\right)^2 + 2D_1\left(\frac{d^2W}{dr^2}\right)\left(\frac{1}{r}\frac{dW}{dr}\right) + D_\theta\left(\frac{1}{r}\frac{dW}{dr}\right)^2\right]r\,dr \tag{2}$$

and

$$T = \pi h\int_a^b \sigma_r\left(\frac{dW}{dr}\right)^2 r\,dr \tag{3}$$

On substituting for σ_r from Eq. (1), T becomes

$$T = -\pi h\frac{p_i}{1-(a/b)^{2k}}\int_a^b\left(\frac{a}{r}\right)^{k+1}\left[1-\left(\frac{r}{b}\right)^{2k}\right]\left(\frac{dW}{dr}\right)^2 r\,dr \tag{4}$$

The problem is to find solutions satisfying the stationary condition[9] of total potential energy, i.e.,

$$\delta(V+T) = 0 \tag{5}$$

In applying the Rayleigh-Ritz method, the axisymmetric mode W is expressed as

$$W = \sum_{i=1}^{\infty} A_i v_i(r) \tag{6}$$

in which $v_1, v_2, v_3, \ldots$ are chosen admissible functions such that W satisfies the relevant geometric boundary conditions. The parameters A_i are to be determined from the stationary condition (5). This process leads to a set of linear, homogeneous, simultaneous equations in the A_i's. For nontrivial solutions, one obtains the characteristic equation for the determination of eigenvalues as

$$\det[V_{mn} - p_i a^2 h T_{mn}] = 0 \tag{7}$$

in which

$$V_{mn} = \int_a^b [D_r v_m'' v_n'' + D_1(v_m'' v_n' + v_m' v_n'')/r + D_\theta v_m' v_n'/r^2]r\,dr \tag{8}$$

and

$$T_{mn} = \int_a^b\left(\frac{a}{r}\right)^{k+1}\left[1-\left(\frac{r}{b}\right)^{2k}\right]v_m' v_n' r\,dr \tag{9}$$

where prime denotes differentiation with respect to r.

In Ref. 4, the preceding direct analysis with simple polynomials as admissible functions v_i was found to lead to ill-conditioned equations and, hence, even to obtain a three-term solution, double-precision arithmetic (25 significant digits) was used. In the present investigations, also, this ill-conditioning nature is found to exist. However detailed numerical investigations have shown that the conditioning of equations improves with decreasing hole size and, in some cases of edge conditions, even single-precision arithmetic is found to be adequate for small hole sizes.

3. Coordinate Transformation

In order to achieve accurate results for large hole sizes, the analysis is modified by means of a coordinate transformation

$$y = (r^2 - a^2)/(b^2 - a^2) \tag{10}$$

which was effectively used in the case of vibration problems.[7]

Table 1 Admissible functions[a]

Boundary condition: Outer	Boundary condition: Inner	Designation	$v_o(r)$	$u_o(y)$
Simply supported	Free	S-F	$(b-r)$	$(1-y)$
Free	Fixed	F-C	$(r-a)^2$	y^2
Fixed	Free	C-F	$(b-r)^2$	$(1-y)^2$
Simply supported	Simply supported	S-S	$(b-r)(r-a)$	$y(1-y)$
Simply supported	Fixed	S-C	$(b-r)(r-a)^2$	$y^2(1-y)$
Fixed	Simply supported	C-S	$(b-r)^2(r-a)$	$y(1-y)^2$
Fixed	Fixed	C-C	$(b-r)^2(r-a)^2$	$y^2(1-y)^2$

[a] $W = v_o(r)[A_1 + A_2 r + A_3 r^2 + \ldots]$ (6).
$W = u_o(y)[B_1 + B_2 y + B_3 y^2 + \ldots]$ (13).

When the transformation (10) is used, the expressions for V and T become

$$V = \frac{2\pi}{b^2-a^2}\int_0^1 \left\{ D_r\left[2(y+c)\frac{d^2W}{dy^2} + \frac{dW}{dy}\right]^2 + 2D_1\left[2(y+c)\frac{d^2W}{dy^2} + \frac{dW}{dy}\right]\frac{dW}{dy} + D_\theta\left(\frac{dW}{dy}\right)^2\right\} dy \tag{11}$$

and

$$T = -2\pi p_i h \frac{c}{1-(a/b)^{2k}}\int_0^1 \left(\frac{y+c}{c}\right)^{(1-k)/2} \times \left[1-\left(\frac{y+c}{1+c}\right)^k\right]\left(\frac{dW}{dy}\right)^2 dy \tag{12}$$

in which

$$c = a^2/(b^2-a^2)$$

Expressing the mode W as

$$W = \sum_{i=1}^{\infty} B_i u_i(y) \tag{13}$$

and proceeding as before, one obtains the characteristic equation as

$$\det\left[J_{mn} - (p_i a^2 h/D_r) I_{mn}\right] = 0 \tag{14}$$

in which $J_{mn} = J_{nm}$, $I_{mn} = I_{nm}$, and these are given by

$$J_{mn} = 4\int_0^1 (y+c)^2 u_m'' u_n'' dy + \left(\frac{D_\theta}{D_r} - 1\right)\int_0^1 u_m' u_n' dy + 2(1+\nu_\theta)\left[(y+c)u_m' u_n'\right]_{y=0}^{y=1} \tag{15}$$

and

$$I_{mn} = \frac{1}{1-(a/b)^{2k}}\int_0^1 \left(\frac{y+c}{c}\right)^{(1-k)/2} \times \left[1 - \left(\frac{y+c}{1+c}\right)^k\right] u_m' u_n' dy \tag{16}$$

where prime denotes differentiation with respect to y.

4. Numerical Results

In the present analysis, the admissible functions $v_i(r)$ and $u_i(y)$ are chosen to be simple polynomials in r and y (see Table 1), as used in the case of vibration problems.[7] Estimates of the critical buckling loads have been obtained for hole sizes 0.1–0.9 in steps of 0.1 and for all the nine combinations of clamped,

Table 2 Critical buckling load parameter ($p_i a^2 h/D$) of isotropic annular plate: $\nu_\theta = \nu_r = \nu = 0.3$

Case[a]	From equation	N[b]	Hole size a/b: 0.1	N	0.2	N	0.3	N	0.4
S-F	(7)	5	1.0281	4	1.1240	3	1.2180	3	1.30964
F-S		6	1.0272	5	1.1238	4	1.2177	4	1.30962
F-F		6	(1.0272)[c]	5	(1.1238)	4	(1.2177)	4	(1.30962)
	(14)	8	1.39	8	1.195	8	1.223	8	1.3098
C-F	(7)	4	1.2193	4	1.6833	4	2.462	4	3.830
		5	1.2191	5	1.6826	5	2.461	5	3.829
		5	(1.2191)	5	(1.6828)	4	(2.462)	4	(3.830)
	(14)	8	1.61	8	1.724	8	2.461	8	3.829
F-C	(7)	6	2.379	4	3.736	4	5.8135	4	9.286
		7	2.378	5	3.732	5	5.8134	5	9.282
		5	(2.385)	4	(3.734)	4	(5.811)	4	(9.280)
	(14)	8	2.91	8	3.85	8	5.824	8	9.282
S-S	(7)	6	2.133	6	4.290	5	8.140	5	15.105
		7	2.132	7	4.288	6	8.132	6	15.102
		6	(2.128)	5	(4.296)	4	(8.22)	4	(15.17)
	(14)	8	2.83	8	4.32	8	8.140	8	15.107
C-S	(7)	6	2.738	6	6.1046	5	12.1896	5	23.3139
		7	2.736	7	6.1038	6	12.1896	6	23.3139
		6	(2.745)	4	(6.16)	3	(12.52)	3	(23.8)
	(14)	8	3.397	8	6.110	8	12.21	8	23.318
S-C	(7)	6	6.0758	6	13.04	6	24.83	6	45.682
		7	6.0758	7	13.03	7	24.82	7	45.678
		5	(6.26)	3	(13.37)	3	...	3	...
	(14)	8	8.58	8	13.53	8	24.83	8	45.697
C-C	(7)	6	7.680	6	17.27	6	33.709	6	62.972
		7	7.678	7	17.26	7	33.705	7	62.972
		3	(8.05)	3	...	3	...	3	...
	(14)	8	10.5	8	17.63	8	33.72	8	63.000

[a] C = clamped, S = simply supported, F = free.
[b] N = number of terms used in computation.
[c] Values in parentheses are obtained using single-precision arithmetic (7 significant digits).

Table 3 Critical buckling load parameter ($p_i a^2 h/D$) of isotropic annular plate: $\nu_\theta = \nu_r = \nu = 0.3$

		Hole size a/b									
Case[a]	From equation	N[b]	0.5	N	0.6	N	0.7	N	0.8	N	0.9
S-F	(14)	5	1.403	4	1.492	4	1.574	3	1.660	3	1.74008
F-S		6	1.400	5	1.488	5	1.574	4	1.658	4	1.74006
F-F		6	(1.401)[c]	5	(1.489)	5	(1.575)	4	(1.658)	4	(1.741)
	(7)	3	(1.400)	3	(1.488)	3	(1.574)	3	(1.658)	3	(1.7402)
C-F	(14)	5	6.392	4	11.68	4	24.30	4	63.998	3	299.30
		6	6.391	5	11.66	5	24.29	5	63.992	4	298.93
		6	(6.391)	5	(11.66)	5	(24.29)	5	(63.992)	4	(298.93)
	(7)	4	6.391	4	11.66	4	24.29	4	63.992	4	298.93
F-C	(14)	5	15.55	5	28.12	4	57.67	4	149.30	4	685.06
		6	15.547	6	28.12	5	57.66	5	149.21	5	684.78
		5	(15.55)	5	(28.12)	5	(57.67)	5	(149.22)	5	(684.78)
	(7)	5	15.548	4	28.13	4	57.68	4	149.25	4	684.9
S-S	(14)	6	28.43	6	56.06	5	122.80	5	333.49	4	1585.4
		8	28.39	8	56.05	6	122.75	6	333.49	5	1584.8
		5	(28.60)	5	(56.20)	6	(122.63)	6	(333.44)	5	(1584.8)
	(7)	5	28.39	5	56.05	5	122.77	5	333.53	4	1585.0
C-S	(14)	6	44.65	6	89.11	5	196.61	5	536.71	4	2558.2
		8	44.62	8	89.11	6	196.61	6	536.65	5	2557.4
		4	(45.44)	5	(89.05)	5	(196.50)	5	(536.72)	5	(2557.9)
	(7)	5	44.63	5	89.12	5	196.63	4	536.92	3	2564
S-C	(14)	6	84.98	6	165.5	6	357.8	5	962.1	5	4522.0
		8	84.82	8	165.3	8	357.75	6	961.2	6	4521.7
		5	...	4	(168)	4	(363)	4	(972)	4	(4555)
	(7)	6	84.81	6	165.3	4	360.5	4	967	3	4683
C-C	(14)	6	118.3	6	231.94	6	504.33	5	1360.2	5	6416.0
		8	118.1	8	231.86	8	504.32	6	1360.2	6	6415.3
		4	(120)	4	(236)	3	(519)	3	(1400)	3	(6574)
	(7)	5	118.1	5	231.84	4	505.1	3	1390	3	...

[a] C = clamped, S = simply supported, F = free.
[b] N = number of terms used in computation.
[c] Values in parentheses are obtained using single-precision arithmetic (7 significant digits).

simply supported, and free edge conditions. The Poisson's ratio ν_θ is fixed at 0.3.

The rigidity parameters D_θ/D_r and D_r/D_θ are varied from 0.3 to 1.0 in steps of 0.1. It is to be noted that the flexural rigidities D_θ and D_r positive imply that the product $(\nu_r \nu_\theta)$ cannot be greater than unity. Hence the ratio $(D_\theta/D_r)^{1/2}$ cannot be less than ν_θ. Mossakowski[2] derived a more restrictive condition that D_θ/D_r cannot be less than ν_θ on the basis that the diameter of a solid circular disk under uniform compression cannot increase. In a general case,[10] none of the six Poisson's ratios of orthotropic materials is greater than unity. This implies that the condition D_θ/D_r cannot be less than ν_θ is valid for annular plates as well.

The buckling loads obtained in the three cases of one edge free and the other either free or simply supported are identical. In fact, it has been established earlier, both mathematically[4] and physically,[5] that all the eigenvalues corresponding to axisymmetric modes in these three cases are identical. Physically, it is shown[5] that shear resultant in the axisymmetric case is zero everywhere if one of the edges is unsupported. This is equivalent to zero support reaction all along any concentric circular support. Hence, in the particular case of both edges free, the eigenvalues and thereby the critical buckling loads are independent of the location of the concentric circle of support.

5. Accuracy of Numerical Results

Estimates of critical buckling loads were obtained for various numbers of terms to study the convergence of solutions. Single-precision (7 significant digits) as well as double-precision arithmetic (16 significant digits) was used to find the influence of rounding errors in computations. All computations were carried out on an IBM 360/44 Digital Computer at the Indian Institute of Science, Bangalore.

In order to assess the relative merits of solving Eqs. (7) and (14), the estimates of the critical buckling load obtained for isotropic plates are presented in Tables 2 and 3 for various orders of approximations.

From Rayleigh's principle, these estimates are upper bounds to the exact values. Since no exact data are available, the data obtained by using double precision arithmetic in computations are presented for two successive orders of approximation such that the maximum error is likely to be in the fourth significant digit only. It can be seen from the number of terms indicated in Tables 2 and 3 that the convergence of solutions generally improves with increasing hole sizes for both direct and modified analysis, and the rate of convergence is much higher in the former analysis. This is perhaps due to the fact that the mode is represented by both odd and even powers of r in the direct analysis, whereas only even powers of r are considered in the modified analysis. An interesting observation is that, in contrast to the common expectation, the convergence of solutions tends to become slower with increasing number of geometric constraints on the edges.

To understand the behavior of ill-conditioning nature of the equations, the estimates obtained by using single precision arithmetic are also presented in Tables 2 and 3 for hole sizes up to 0.4 in the case of direct analysis and for hole sizes beyond 0.4 in the case of modified analysis along with the number of terms used in the analyses. It is to be noted that the estimates are considerably affected by rounding errors for a higher order approximation in cases where the number of terms indicated is less than the corresponding number of terms used to obtain estimates by double-precision arithmetic.

Table 4 Comparison of estimates of critical buckling load parameter ($p_i a^2 h/D$) with values obtained by iterative procedure (Ref. 6)[a]

	Hole size a/b					
Analysis	N	0.1	N	0.2	N	0.5
Present analysis	7	2.397[b]	7	3.768[b]	7	15.695[c]
Ref. 6		2.33		3.63		15.0

[a] Isotropic annular plate: F-C case, $\nu_\theta = \nu_r = \nu = \frac{1}{3}$.
[b] From Eqn. (7).
[c] From Eq. (14).

In the direct analysis, the conditioning of the equations generally deteriorates with increasing hole size, particularly in the four cases of one edge simply supported and the other either simply supported or clamped. For hole sizes beyond 0.5, double-precision arithmetic is required in all cases except the S-F case. Influence of rounding errors becomes particularly predominant in the C-C case with increasing hole size and, in fact, for a hole size 0.9, even double precision-arithmetic is not adequate to obtain a three-term solution. In the abovementioned situations, the modified analysis is found to be very suitable. In fact, for hole sizes beyond 0.5, even single-precision arithmetic is found to be quite adequate in all cases, although the convergence is relatively slow compared to the direct analysis, as mentioned earlier. However, the modified analysis is found to be inconvenient for small hole sizes, as in the vibration problems.[7]

In Table 4, the estimates by the present analysis are compared with those obtained by Rózsa[6] in the F-C case for $\nu = \frac{1}{3}$. When compared with the corresponding estimates for $\nu = 0.3$, the present estimates are slightly higher, whereas the estimates of Ref. 6 are much lower. The former is in confirmity with the corresponding behavior in the case of free flexural vibrations.[11] Moreover, the present estimates by seven-term approximation are highly accurate. Hence, one concludes that the estimated values in Ref. 6 are of relatively low accuracy. One may, however, note that these values obtained by the iterative procedure correspond to the second stage of approximation only.

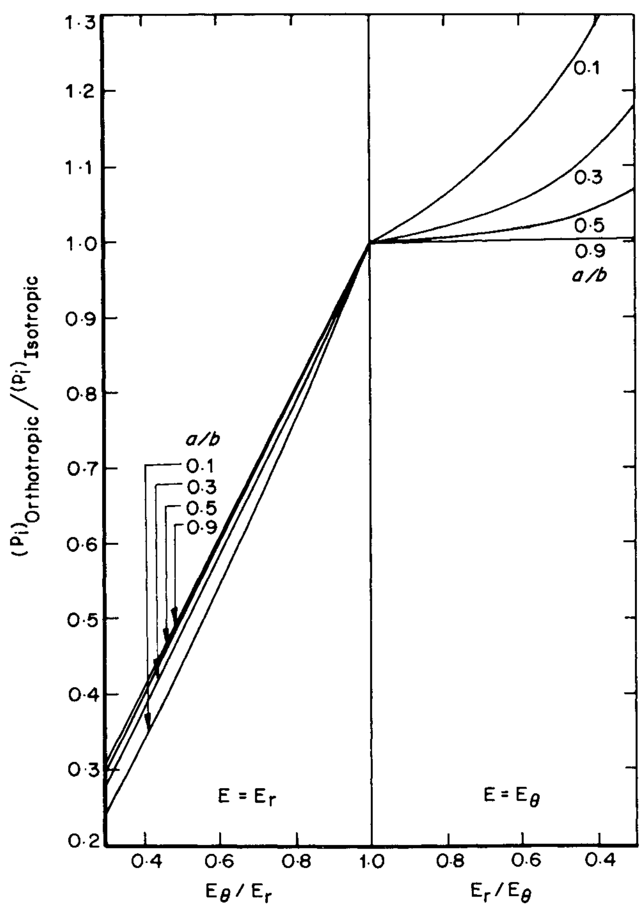

Fig. 2 Variation of critical buckling load parameter with rigidity ratio: S-F case (same also for F-S and F-F cases[4]).

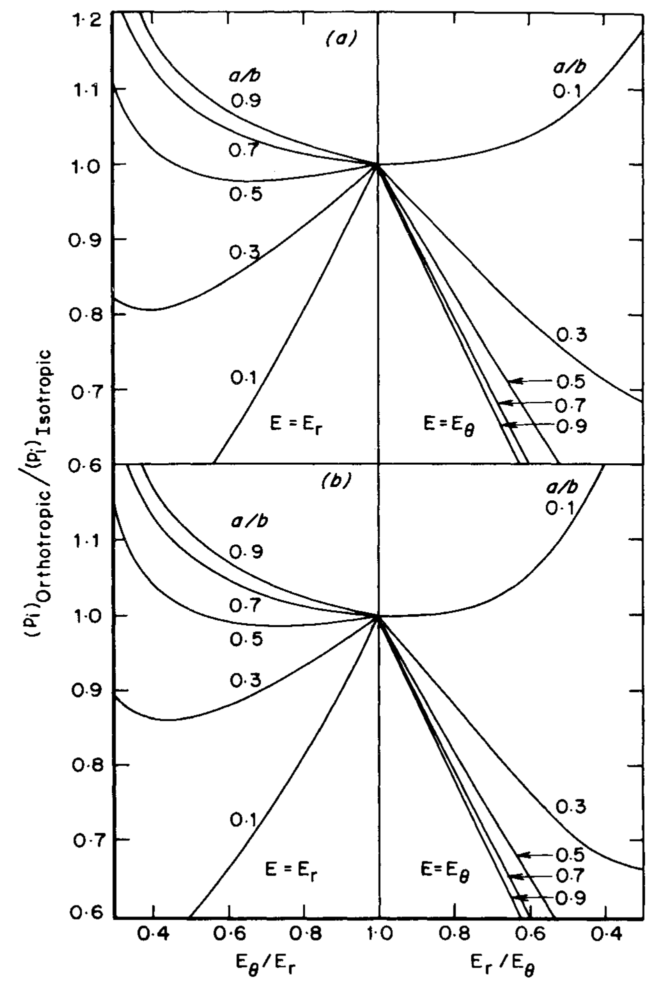

Fig. 3 Variation of critical buckling load parameter with rigidity ratio: a) C-F case, b) F-C case.

6. Comparison with Isotropic Plates

The effects of the rigidity ratios E_θ/E_r and E_r/E_θ on the buckling loads of orthotropic plates relative to those of the corresponding isotropic plates ($\nu = 0.3 = \nu_\theta$) are shown in Figs. 2–5. The curves in these figures exhibit some interesting features noted below.

In the case of $E_r = E$ and $E_\theta < E_r$, the buckling load of orthotropic plate decreases with decrease in E_θ in the S-F case (Fig. 2) and, hence, in F-S and F-F cases[4] for all hole sizes and for a hole size 0.1 in the C-F and F-C cases (Fig. 3). For hole sizes 0.3 and 0.5 in the C-F and F-C cases and for hole sizes up to 0.3 in S-C, C-S, S-S, and C-C cases (Figs. 4 and 5), the buckling load decreases initially up to a certain minimum value and then increases with decreasing E_θ. For hole sizes beyond 0.7 in the C-F and F-C cases and for hole sizes beyond 0.5 in the remaining four cases, the buckling load continuously increases with decrease in E_θ.

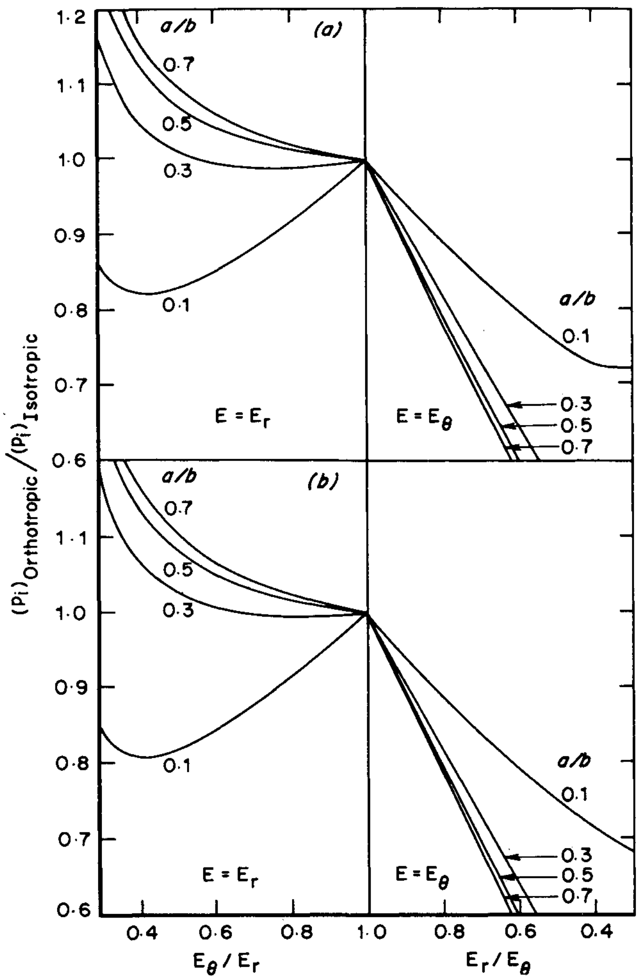

Fig. 4 Variation of critical buckling load parameter with rigidity ratio: a) S-C case, b) C-S case.

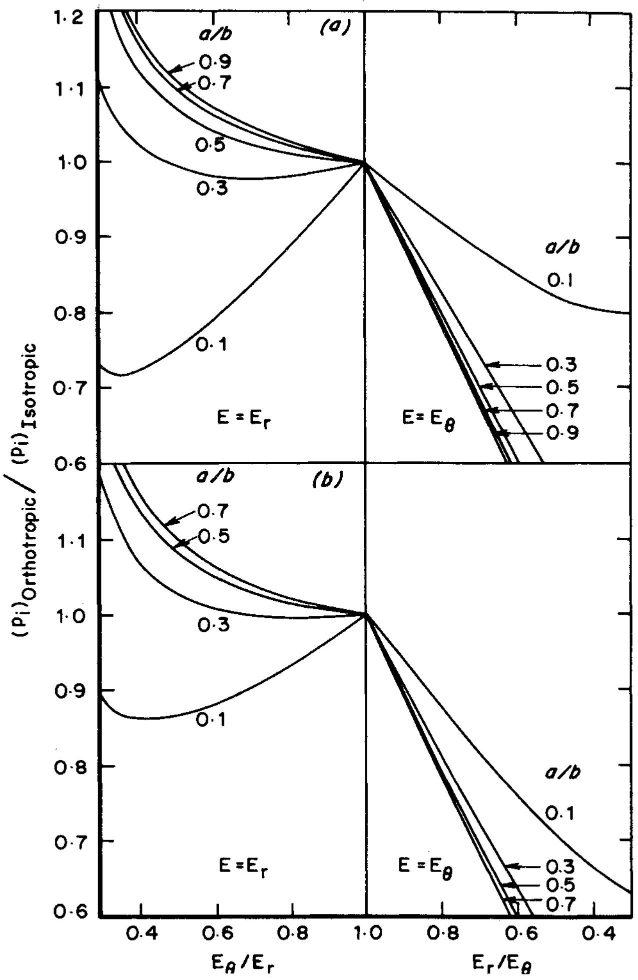

Fig. 5 Variation of critical buckling load parameter with rigidity ratio: a) S-S case, b) C-C case.

In the case of $E_\theta = E$ and $E_r < E_\theta$, the plate behavior is found to be more or less complementary to the behavior previously discussed for all hole sizes and edge conditions.

7. Conclusions

The direct application of the classical Rayleigh-Ritz method with simple polynomials as admissible functions generally has been found to be satisfactory, to obtain critical buckling loads. However, it is found that the conditioning of equations deteriorates with increasing hole sizes. For hole sizes beyond 0.5, use of double-precision arithmetic in computations is required in all cases except the S-F case, and even this is found to be inadequate for hole sizes beyond 0.8 in the two cases of C-C and S-C. In such situations, it is shown that the method in conjunction with the coordinate transformation is convenient to obtain accurate estimates, although the rate of convergence is relatively slow in comparison with the direct analysis.

A comparison of buckling loads of orthotropic plates with those of isotropic plates has brought out some interesting features.

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